Amplification of light in a glass ball suspended within a gainy medium

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We discuss total internal reflection (TIR) from an interface between glass and active gaseous media and propose an experiment for strong light amplification.

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Light reflection from an interface between two media is determined by the wave equation and the boundary conditions, which follow from Maxwell's equations. The wave equations for electric, $\vec{E}$ and magnetic, $\vec{H}$, fields in a homogeneous medium with constant μ and ϵ are

$$\Delta\vec{E}(\vec{r},t) = -\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t),\ \ \Delta\vec{H}(\vec{r},t) = -\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{H}(\vec{r},t). \tag{1}$$

Both equations have plain wave solutions

$$\vec{E}(\vec{r},t) = \vec{E}\exp(i\vec{k}\vec{r}-i\omega t),\ \ \vec{H}(\vec{r},t) = \vec{H}\exp(i\vec{k}\vec{r}-i\omega t), \tag{2}$$

where $k^2 = \epsilon\mu k_0^2$, $k_0 = \omega/c$ and c is the speed of light in vacuum. The fields $\vec{E}$ and $\vec{H}$ are not independent, but are related according to

$$\vec{H} = \frac{1}{\mu\omega}[\vec{k}\times\vec{E}], \qquad \vec{E} = -\frac{1}{\epsilon\omega}[\vec{k}\times\vec{H}]. \tag{3}$$

If $|\vec{E}| = 1$, then the length of $\vec{H}$ is $|\vec{H}| = \sqrt{\varepsilon/\mu} = 1/Z$, where $Z = \sqrt{\mu/\varepsilon}$ is the medium impedance.

If space consists of two halves with different $\epsilon_{1,2}$ and $\mu_{1,2}$, the wave equations Eq. (1) are different for each half, and their solutions should be matched at the interface. The matching conditions follow from the Maxwell equations. They require continuity of the components

$\vec{E}_\parallel(\vec{r},t)$, $\vec{H}_\parallel(\vec{r},t)$ parallel to the interface, and $\epsilon(\vec{n}\cdot\vec{E}(\vec{r},t))$, $\mu(\vec{n}\cdot\vec{H}(\vec{r},t))$, perpendicular to it, where $\vec{n}$ is a unit normal vector. The wave function in presence of the interface is

$$\begin{aligned}&\vec{\psi}(\vec{r},t) =\\ &\Theta(z<0)\Big(\exp(i\vec{k}_1\vec{r}-i\omega t)\vec{\psi}_1+\exp(i\vec{k}_r\vec{r}-i\omega t)\vec{\psi}_r\rho\Big)\\ &+\Theta(z>0)\exp(i\vec{k}_2\vec{r}-i\omega t)\vec{\psi}_2\tau,\end{aligned} \tag{4}$$

where the term $\exp(i\vec{k}_1\vec{r}-i\omega t)\vec{\psi}_1$ with the wave vector $\vec{k}_1$ describes the plain wave incident on the interface from medium 1, factors $\vec{\psi}_i=\vec{E}_i+\vec{H}_i$ $(i=1,r,2)$ denote sum of electric and magnetic polarization vectors, $\vec{k}_{r,2}$ are wave vectors of the reflected and transmitted waves, ρ, τ are the reflection and transmission amplitudes respectively, and $\Theta(z)$ is the step function, which is equal to unity when inequality in its argument is satisfied, and to zero otherwise.

The wave vectors $\vec{k}_{r,2}$ are completely determined by $\vec{k}_1$. They are determined uniquely by the constants ϵ_i, μ_i, and by the fact that $k_0=\omega/c$ and the part $\vec{k}_\parallel$ of the wave vectors parallel the interface must be identical for all the waves. In the following we assume that the medium 1 is lossless, i.e. $\epsilon_1\mu_1$ is real, therefore all the components of $\vec{k}_1$ are also real.

The normal component $k_{2\perp}$ of the refracted wave is

$$k_{2\perp}=\sqrt{\epsilon_2\mu_2k_0^2-\vec{k}_\parallel^2}=\sqrt{k_{1\perp}^2-(\epsilon_1\mu_1-\epsilon_2\mu_2)k_0^2}, \tag{5}$$

or it can be represented as

$$k_{2\perp}=\sqrt{\epsilon k_1^2-\vec{k}_\parallel^2}=\sqrt{n^2k_1^2-\vec{k}_\parallel^2}, \tag{6}$$

where $n=\sqrt{\epsilon}$ is the refractive index, and we introduced relative permittivity $\epsilon=\epsilon_2\mu_2/\epsilon_1\mu_1$.

The amplitudes ρ and τ are well known from textbooks (see [1], for example). They are calculated differently for TE-wave when the incident electric field is polarized perpendicularly to the plane of incidence (or parallel to the interface, and is typically denoted as $\vec{E}_s$), and for TH-field when the incident electric field is polarized in the plane of incidence ($\vec{E}_p$). For both of this cases we have well known Fresnel formulas

$$\rho_s=\frac{\mu_2k_{1\perp}-\mu_1k_{2\perp}}{\mu_2k_{1\perp}+\mu_1k_{2\perp}},\qquad \rho_p=\frac{\epsilon_2k_{1\perp}-\epsilon_1k_{2\perp}}{\epsilon_2k_{1\perp}+\epsilon_1k_{2\perp}}, \tag{7}$$

and $\tau_{s,p}=1+\rho_{s,p}$. In the following we for simplicity assume that $\mu_2=\mu_1=1$.

From Eq. (6) it follows that for lossless media when $0<\epsilon<1$ is real, the incident wave, for which $\vec{k}_\parallel$ is within $nk_1\le|\vec{k}_\parallel|\le k_1$, is totally reflected from the interface. This happens because

$$k_{2\perp}=iK''_{2\perp}\equiv i\sqrt{k_\parallel^2-\epsilon k_1^2}, \tag{8}$$

thus the factor $\exp(ik_{2\perp}z) = \exp(-K''_{2\perp}z)$ of the wave $\exp(i\vec{k}_2\vec{r})$ exponentially decays, i.e. the refracted wave becomes an evanescent one. Therefore, the energy does not flow inside the medium 2, and due to the energy conservation it must be totally reflected into medium 1.

If the medium 2 is lossy or gainy, the constant ϵ is a complex quantity $\epsilon = \epsilon' \pm i\epsilon''$, with positive ϵ' and ϵ''. In this case outside the total internal reflection (TIR) region ($|\vec{k}_\parallel|^2 \ll \epsilon' k_1^2$) we have $k_{2\perp} = k'_{2\perp} \pm ik''_{2\perp}$, where for small ϵ'' ($\epsilon'' k_1^2 \ll \epsilon' k_1^2 - |\vec{k}_\parallel|^2$)

$$k'_{2\perp} \approx \sqrt{\epsilon' k_1^2 - |\vec{k}_\parallel|^2}, \quad k''_{2\perp} \approx \epsilon'' \frac{k_1^2}{2k'_{2\perp}}. \tag{9}$$

In the TIR regime, $k'_{2\perp}$ in Eq. 9 transforms into $iK''_{2\perp}$, where $K''_{2\perp} \approx \sqrt{|\vec{k}_\parallel|^2 - \epsilon' k_1^2}$, and $k''_{2\perp}$ transforms to

$$k''_{2\perp} \to -iK'_{2\perp} = \epsilon'' \frac{k_1^2}{2iK''_{2\perp}}. \tag{10}$$

Therefore, at TIR $k_{2\perp} = \pm K'_{2\perp} + iK''_{2\perp}$, where

$$K'_{2\perp} = \epsilon'' \frac{k_1^2}{2K''_{2\perp}}, \qquad K''_{2\perp} \approx \sqrt{|\vec{k}_\parallel|^2 - \epsilon' k_1^2}. \tag{11}$$

The '+' sign before imaginary part $iK''_{2\perp}$ determines exponential decay of the refracted wave away from the interface for both lossy and gainy media cases. However the real part, $K'_{2\perp}$ has opposite signs for lossy and gainy cases.

The reflection amplitudes Eq. (7) at TIR become

$$\rho_s = \frac{k_{1\perp} - iK''_{2\perp} \mp K'_{2\perp}}{k_{1\perp} + iK''_{2\perp} \pm K'_{2\perp}}, \tag{12}$$

$$\rho_p = \frac{\epsilon_2 k_{1\perp} - \epsilon_1(iK''_{2\perp} \pm K'_{2\perp})}{\epsilon_2 k_{1\perp} + \epsilon_1(iK''_{2\perp} \pm K'_{2\perp})}. \tag{13}$$

The positive value of $K'_{2\perp}$ for lossy medium means that the reflection coefficient in TIR is less than one, because part of the energy flux proportional to $K'_{2\perp}$ enters the medium 2 and decays there. The negative value of $K'_{2\perp}$ for gainy medium means that the reflection coefficient in TIR is larger than one, because part of the energy flux proportional to $K'_{2\perp}$, exits the medium 2 and adds to the TIR wave.

It is necessary to mention here the work [2] (see also critics in [3]), where the author claims that in the case of TIR from a gainy medium the wave vector inside the gainy medium has opposite sign: $k_{2\perp} = K'_{2\perp} - iK''_{2\perp}$, and the reflection coefficient at TIR $|\rho_{s,p}|^2$ is less than unity. If that was the case, the wave function inside the gainy medium would increase proportionally to $\exp(K''_{2\perp}z)$ even for tiniest gain. Since $K''_{2\perp} \sim 1/\lambda$ (see Eq. (11)) then at a distance 1 mm from the interface inside the gainy medium (for $\lambda \sim 1000$ nm) the intensity of the field would be $e^{2000} \approx 10^{860}$ times larger than the intensity of the incident light, surpassing all astronomical numbers.

With the correct sign $k_{2\perp} = -K'_{2\perp} + iK''_{2\perp}$ the reflection coefficient at TIR from a gainy medium is larger than one, and it increases with the gain. The photons induced by the incident wave cannot propagate inside the gainy material for the same reasons the incident photons do not. They can only tunnel from the gainy medium into the first medium, and coherently add to the reflected primary photons, thus increasing the reflected flux. The larger is the gain, the larger is the probability of such a process.

We want also to note here that the widely spread belief that the energy flux is given by the Pointing vector $\vec{J} = [\vec{E} \times \vec{H}]$ is not correct. The energy flux is given by

$$\vec{J} = c\frac{\vec{k}}{k}\frac{\epsilon E^2 + \mu H^2}{8\pi}, \tag{14}$$

i.e. it is equal to the energy density times the light speed, and it has direction along the wave vector $\vec{k}$. For a plain wave this definition coincides with the Pointing vector. However the latter can be defined for wider varieties of vectors $\vec{E}$ and $\vec{H}$, including stationary fields and evanescent waves for which the Pointing vector has no relation to the energy flux.

The experiment for strong enhancement of the light trapped in a glass sphere

The increase of the reflection coefficient at TIR from a gainy medium can be used to develop a curious experiment for storage and amplification of light. Imagine a glass sphere with a coupler P, as shown in Fig. 1. The sphere has thin walls (it is also possible to use a homogeneous glass sphere) and is surrounded by an excited gas (or other active media). The ray of light, shown by the thin line, enters the glass walls through the coupler and then undergoes TIR multiple times. At every reflection the light is amplified according to the analysis in the previous Section. At the end the ray escapes the sphere, as shown by the thick line. The amplification depends on the number of the reflections and on the gain coefficient of the active medium. The number of the reflections is very sensitive to the angle of the incident ray. If the overall amplification is sufficiently high, the glass will melt into a liquid bubble with thin skin filled with the light, similar to the ball lightning [4].

We can estimate the magnitude of the light enhancement in such a sphere. Assume that for the active medium $\epsilon_2 \approx 1 - i\alpha$, and ϵ_1 of the glass is real. For TE-mode, the reflection amplitude at TIR according to Eq. 12 and 11 can be written as

$$\rho_s \approx \frac{k_{1\perp} - iK_{2\perp} + \alpha k_0^2/2K_{2\perp}}{k_{1\perp} + iK_{2\perp} - \alpha k_0^2/2K_{2\perp}}, \tag{15}$$

where $K_{2\perp} = \sqrt{(\epsilon_1 - 1)k_0^2 - k_{1\perp}^2}$, and the approximation is valid for small α. Similarly we can write equation for ρ_p. For estimating purposes we can assume that in both cases the light is amplified by approximately $1 + 2\alpha$ at each reflection.

Let's consider a sphere of radius $R = 10$ cm submerged into an active medium with small α. The energy I inside it increases with number N of collisions with the walls $\propto (1+2\alpha)^N \approx$

$\exp(2N\alpha)$. The N can be represented via time t as t/t_1, where $t_1 = 2R\sin\theta\sqrt{\epsilon_1}/c$ is the time between two consecutive collisions of photons with the walls, and θ is the grazing angle with surface at the collision point. Therefore $I(t) = I_0\exp(t/\tau)$, where $\tau = t_1/2\alpha = R\sin\theta\sqrt{\epsilon_1}/c$, and I_0 is initial energy of the incident ray. For $\theta = 0.1$ and $\alpha = 10^{-7}$ we get $1/\tau \approx 3\times 10^3$ s^{-1}. Therefore, for $I_0 = 10^{-19}$ J, the energy I after 20 ms reaches 10 MJ.

The following analysis is used to estimate α. The amplification of a laser beam along a path l inside a gainy media is $\exp(2k''l)$, where k'' is the imaginary part of the wave number, and $g = 2k''$ is called the gain coefficient. In a medium with $\epsilon = 1 - i\alpha$, the gain coefficient is $g \approx \alpha k = 2\pi\alpha/\lambda$, where λ is the wavelength. For N_2,CO_2 gas lasers, the gain coefficient is approximately 10^{-2} cm^{-1} [5]. For $\lambda/2\pi \simeq 10^{-4}$ cm we obtain $\alpha = 10^{-6}$.

In the past, many experiments were performed with the whispering gallery mode resonators (WGMR) of small dimensions (~ 1 mm) and large Q-factors (up to $Q\sim 10^{10}$) [6], where light undergoes large number $N \sim Q$ total internal reflections. In a larger sphere submerged into an active medium with $\alpha \sim 10^{-7}$, the Q-factor can also be large, resulting in enormous accumulated energy. The stored photons will heat and melt the resonator, but the electrostriction forces will hold the melted substance together. One can expect to see many interesting nonlinear phenomena in such systems.

Spherical harmonics analysis

Spherical harmonics are used in [6,7] and many other works dealing with microspheres, and therefore it is important to analyze their applicability in our case. The TE or TH fields inside and outside the sphere are represented by functions like

$$\psi(\vec{r},t) = \exp(-i\omega t)Y_{l,m}(\theta,\phi)F_l(r), \tag{16}$$

where $Y_{l,m}(\theta,\phi)$ are the spherical harmonics, and the radial function $F_l(r)$ can be represented as

$$F_l(r) = \Theta(r<R)\frac{j_l(nk_0r)}{j_l(nk_0R)} + \Theta(r>R)\frac{h_l^{(1)}(n'k_0r)}{h_l^{(1)}(n'k_0R)}, \tag{17}$$

where $j_l(kr)$, $h_l^{(1)}(k'r)$ are spherical Bessel and Hankel functions respectively, n and n' are the refractive indices inside and outside the sphere, $k_0 = \omega/c$, and the factor before $h_l^{(1)}(n'k_0r)$ ensures continuity of the function $F_l(r)$ at $r = R$. The second boundary condition, requiring continuity of the radial derivative, determines values of k_0 for which solution Eq. (17) is possible.

However such an approach is not appropriate for the trapped light in our expriment, because the outside function must be evanescent, while spherical Hankel functions are not. To describe the trapped light in the whispering gallery (WG) mode, which is distributed closely to the sphere radius R and corresponds to $l \gg 1$, we can use the expansion [8]

$$\frac{l^2}{r^2} \approx \frac{l^2}{R^2} - 2(r-R)\frac{l^2}{R^3}, \tag{18}$$

and treat the linear term as a perturbation. In the simplest approximation the radial equation then becomes

$$\left(\frac{d^2}{dr^2} + \epsilon k^2 - \frac{l^2}{R^2}\right) F_l(r) = 0. \tag{19}$$

Its solution is

$$F_l(r) = \Theta(r<R)\frac{\sin(k_r r)}{\sin(k_r R)} + \Theta(r>R)e^{-K_r(r-R)}, \tag{20}$$

where $k_r \approx \sqrt{\epsilon k_0^2 - l^2/R^2}$ and $K_r = \sqrt{(\epsilon-\epsilon')k_0^2 - k_r^2}$. To obtain the WG mode we must have $\epsilon' < \epsilon$ and sufficiently large l, in order for the arguments of both square roots to be positive.

The second boundary condition gives limitations (or quantization) for k_0^2. If the medium outside the sphere is gainy, then ϵ' contains negative imaginary part $-i\epsilon''$, and the second boundary condition makes k_0^2 or ω complex numbers with positive imaginary part $i\omega''$. It means that the factor $\exp(-i\omega t)$ in Eq. (16) results in the exponential growth $\sim \exp(\omega'' t)$.

The approximation Eq. (19) is equivalent to reflection in plane geometry, while the form Eq. (16) of the wave function means that the distribution of field in the sphere is periodic, or that all the rays in the WG mode are closed. In general, it is not so for the rays shown in Fig. 1. Therefore, when $Rk_0 \gg 1$, the analysis of the trapped light in WG mode with spherical harmonics is not appropriate and is reduced to the plane wave approach.

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References

1. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier Butterworth-Heinemann, 2004)
2. A.E. Siegman, "Fresnel Refl ection, Lenserf Reflection and Evanescent Gain," OPN **21,** 38–45 (2010).
3. F.V. Ignatovich and V.K. Ignatovich, "On Fresnel Reflection and Evanescent Gain," OPN **21,** 6 (2010).
4. V.K.Ignatovich, "The Ball Lightning," Las. Phys. **2,** 991–996 (1992).
5. E.W.McDaniel, W.L.Higham (ed-s), *Applied atomic collision physics* (Academic Press, 1982) Volume 3, Chapter 8, Figure 2.

6. J. U. Fürst, D. V. Strekalov, D. Elser, M. Lassen, U. L. Andersen, C. Marquardt, and G. Leuchs, “Naturally Phase-Matched Second-Harmonic Generation in a Whispering-Gallery-Mode Resonator,” Phys. Rev. Lett. **104,** 153901 (2010).
7. J. R. Buck and H. J. Kimble, “Optimal sizes of dielectric microspheres for cavity QED with strong coupling,” Phys. Rev. A 67, 033806 (2003)
8. Vladimir S. Ilchenko, Anatoliy A. Savchenkov, Andrey B. Matsko, and Lute Maleki, “Dispersion compensation in whispering-gallery modes,” J. Opt. Soc. Am. A 20, 157 (2003).

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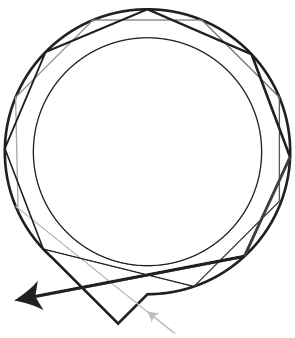

Fig. 1. Schematic of the experiment for multiple TIR off gainy medium. BALL.EPS